Beyond Locality-Sensitive Hashing

Alexandr Andoni\textsuperscript{1}, Piotr Indyk\textsuperscript{2}, Huy L. Nguy\`en\textsuperscript{3}, Ilya Razenshteyn\textsuperscript{2}

\textsuperscript{1}Microsoft Research SVC

\textsuperscript{2}MIT, CSAIL

\textsuperscript{3}Princeton

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The Near Neighbor Problem

- Let $P$ be an $n$-point subset of a metric $(X, D)$, $r > 0$
- For $q \in X$ find any $p \in P$ with $D(p, q) \leq r$
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- Hard, if $(X, D)$ is high-dimensional (space or query time is exponential in the dimension)
The Approximate Near Neighbor Problem (ANN)

- Let $P$ be an $n$-point subset of a metric $(X, D)$, $r > 0$, $c > 1$
- For $q \in X$ find any $p' \in P$ with $D(p', q) \leq cr$, provided that there exists $p \in P$ with $D(p, q) \leq r$
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Exponential dependence on the dimension:
(Arya, Mount 1993), (Meister 1993), (Clarkson 1994),
(Arya, Mount, Netanyahu, Silverman, We, 1998), (Kleinberg, 1997),
(Har-Peled 2002)

Polynomial dependence on the dimension:
(Indyk, Motwani 1998), (Kushilevitz, Ostrovsky, Rabani 1998),
(Indyk 1998), (Indyk 2001), (Gionis, Indyk, Motwani 1999),
(Charikar 2002), (Datar, Immorlica, Indyk, Mirrokni 2004),
(Chakrabarti, Regev 2004), (Panigrahy 2006), (Ailon, Chazelle 2006),
(Andoni, Indyk 2006), (Indyk, Kapralov 2013), (Nguyễn 2013)
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A hash family \( \mathcal{H} \) on \((X, D)\) is \( (r, cr, p_1, p_2) \)-sensitive, if for every \( p, q \in X \):
  - if \( D(p, q) \leq r \), then \( \Pr_{h \sim \mathcal{H}}[h(p) = h(q)] \geq p_1 \);
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Known LSH constructions

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(Motwani, Naor, Panigrahy 2007), (O’Donnell, Wu, Zhou 2011),
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Bounds on $\rho = \ln(1/p_1)/\ln(1/p_2)$ for various spaces:

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This work: ANN in space $O(n^{1+\tau} + nd)$ and time $O(dn^\tau)$, where

- $\tau \leq \frac{7}{8c} + O\left(\frac{1}{c^{3/2}}\right) + o(1)$ for $\ell_1$
- $\tau \leq \frac{7}{8c^2} + O\left(\frac{1}{c^{3}}\right) + o(1)$ for $\ell_2$

The first improvement upon (Indyk, Motwani 1998) for $\ell_1$ and
(Andoni, Indyk 2006) for $\ell_2$!
The main idea

- LSH is oblivious, can we construct a hash family that would depend on the data?

- Too strong! Enough to satisfy these for $p \in P$ and $q \in X$.

Parallels with practice!

- PCA trees (Sproull 1991), (McNames 2001), (Verma, Kpotufe, Dasgupta 2009)
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- Partition $P$ into low-diameter clusters (of diameter $O(cr)$)
- Improve upon $1/c^2$ for the low-diameter case
The low-diameter case

- All points and queries are on a sphere of radius $O(cr)$
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  \[ \rho = \frac{\ln(1/p_1)}{\ln(1/p_2)} \leq \frac{1 - \Omega(1)}{c^2} \]
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From LSH to ANN: the basic reduction

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Hash the dataset $P$ using a concatenation of $k$ functions from $\mathcal{H}$: $x \mapsto (h_1(x), h_2(x), \ldots, h_k(x))$
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The optimal choice of $k$ leads to the need in $n^\rho$ independent hash tables

Overall: $n^{1+\rho}$ space, $n^\rho$ query time
Partition space somewhat coarsely (using smaller $k$ than before)
From LSH to ANN: two-level hashing

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Reflections

Why is this family data-dependent?

- Use a point from \( P \) as a center for an inner hash table
- If \( q \in X \) is far from the center of the outer bin, then we cannot handle it (but we do not care about this case)
Jung’s theorem: any set of diameter $D$ lies in a ball of radius $D/\sqrt{2}$

For each bin find a smallest enclosing ball and hash wrt its center
Smallest enclosing balls

- Jung’s theorem: any set of diameter $D$ lies in a ball of radius $D/\sqrt{2}$
- For each bin find a smallest enclosing ball and hash wrt its center
- Careful analysis leads to

$$\rho \leq \frac{7}{8c^2} + O\left(\frac{1}{c^3}\right) + o_c(1)$$
Can embed $\ell_1$ into $\ell_2$-squared, which gives an algorithm with

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for $\ell_1$ (in particular, Hamming distance for binary strings)
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- Instead of two-level hashing, can consider many levels; preliminary computations give

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- Using this multilevel partitioning can improve known constructions for spanners for subsets of $\ell_1$ and $\ell_2$
  (upon (Har-Peled, Indyk, Sidiropoulos 2013))
Able to overcome the LSH barrier for the case of $\ell_1$ and $\ell_2$ using data-dependent hashing

Can one improve our bounds?
Conclusions and open problems

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- For a certain random instance can achieve $1/(2c)$ and $1/(2c^2)$, which is tight for the data-dependent hashing by (Motwani, Naor, Panigrahy 2007)

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- Can one improve the bound for this random instance further? (Looks hard!)

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