Nonlinear Dimension Reduction via Outer Bi-Lipschitz Extensions

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1 Definitions

Lipschitz Extension

- A map $f: X \to Y$ is ***C*-Lipschitz** if for all $x, x' \in X$:

$$d_Y(f(x), f(x')) \le C \cdot d_X(x, x')$$

- **A Lipschitz extension** of a C-Lipschitz map $f: A \to Y$, where $A \subseteq X$, is a C'-Lipschitz map $f': X \to Y$ such that $f(x) = f'(x)$ for all $x \in A$.
- **Kirszbraun theorem:** For $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, there is always an extension with $C' = C$: every map $f: A \to \mathbb{R}^m$ can be extended to the whole $\mathbb{R}^n$ keeping the same Lipschitz constant.

Bi-Lipschitz Outer-Extension

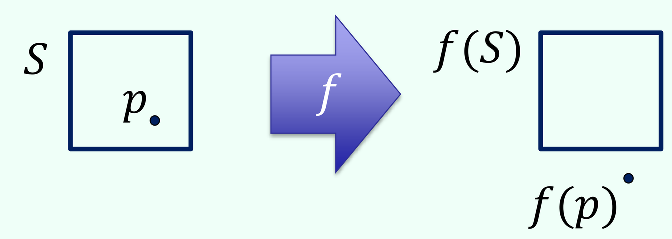
- A map $f: X \to Y$ is ***D*-bi-Lipschitz** or **of distortion *D*** if for some λ and all $x, x' \in X$:

$$\lambda \cdot d_X(x, x') \le d_Y(f(x), f(x')) \le D \cdot \lambda \cdot d_X(x, x')$$

- Consider **Bi-Lipschitz extensions** of bi-Lipschitz maps

Is there an analogue of the Kirszbraun theorem for bi-Lipschitz maps?

- **No direct analogue!**
- **Allow f' to use extra coordinates!**



- Map $f': X \to \mathbb{R}^{m+n}$ **is a *D'*-outer bi-Lipschitz extension** of a D-bi-Lipschitz map $f: A \to Y$, where $A \subset X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, if
  - f' is an outer extension of f: for every $x \in A$

$$f'(x) = f(x) \oplus (0, \dots 0)$$

  - f' is D' bi-Lipschitz

2 Results

Consider a D-bi-Lipschitz map $f: A \to \mathbb{R}^m$ where $A \subset \mathbb{R}^n$,

Analogue of the Kirszbraun theorem:

- ✓ There is an outer bi-Lipschitz extension $f': \mathbb{R}^n \to \mathbb{R}^{m+n}$ of f with distortion $3D$

Near isometric maps: assume that f has distortion $1 + \epsilon$

- ✓ f can be extended by one point to $f': A \cup \{u\} \to \mathbb{R}^{m+1}$ with distortion $1 + O(\sqrt{\epsilon})$. This bound is **tight.**
- ✓ 1-dimensional case: if $n = m = 1$, f can be extended to $f': \mathbb{R} \to \mathbb{R}^2$, with distortion $1 + \Omega\left(\frac{1}{\log^2 1/\epsilon}\right)$. This bound is **tight**.
- ○ **Open Problem:** complete the picture for **higher dimensions** and **extensions by more than a single point**.

3 Applications to Dimensionality Reduction

Prioritized Johnson-Lindenstrauss

Input:

- a set of n points in $\mathbb{R}^d$
- a ranking π on them

Goal: reduce the dimension s.t.

$$f(x) \in \mathbb{R}^{g(r)} \subset \mathbb{R}^{c \log n}$$

where r is the rank of x and g is polylogorithmic

	Distortion	#Non-zero
[EFN'15]*	$O_\epsilon(\log^4 k)$	$O_\epsilon(\log^{4+\epsilon} k)$
This work	$O(\log \log k)$	$\frac{\log^{3+\epsilon} k}{\epsilon^2}$
Open Problem	$(1+\epsilon)$	$\frac{\log k}{\epsilon^2}$

* Elkin, Filtser, and Neiman. Prioritized metric structures and embedding, 2015

Terminal Dimension Reduction

Input: a set $X \subset \mathbb{R}^d$ of n terminals

Goal: find a map $f: \mathbb{R}^d \to \mathbb{R}^{d'}$ s.t. for any $p \in \mathbb{R}^d$ and any terminal $x \in X$,

$$\|x - p\| \le \|f(x) - f(p)\| \le D \cdot \|x - p\|$$

	Distortion	Dimension d'
[EFN'17]*	$O(1)$	$O(\log \lvert X \rvert)$
This work	$1+\epsilon$	$\frac{\log n}{\epsilon^4}$
Open Problem	$1+\epsilon$	$\frac{\log n}{\epsilon^2}$

* Elkin, Filtser, and Neiman. Terminal Embeddings, 2017

4 Analogue of Kirszbraun

Let

- $f(x): A \to \mathbb{R}^m$ be our map with distortion D
- $g = f^{-1}: f(A) \to \mathbb{R}^n$ be its inverse
- $\tilde{f}(x): \mathbb{R}^n \to \mathbb{R}^m$ be the Lipschitz extension of f
- $\tilde{g}(x): \mathbb{R}^m \to \mathbb{R}^n$ be the Lipschitz extension of g

- Assume $\|f\|_{Lip} \le 1$
- Thus $\|g\|_{Lip} \le D$
- By Kirszbraun $\|\tilde{f}\|_{Lip} \le 1$
- By Kirszbraun $\|\tilde{g}\|_{Lip} \le D$

Our bi-Lipschitz extension:

$$f'(x) = \tilde{f}(x) \oplus h(x) \quad \text{where} \quad h(x) = \frac{\tilde{g}(\tilde{f}(x)) - x}{\sqrt{2}D}$$

- ✓ f' is from $\mathbb{R}^n$ to $\mathbb{R}^{n+m}$
- ✓ For $x \in A$, $f(x) = f'(x)$
  - $\tilde{f}(x) = f(x)$
  - $h(x) = 0$ as $\tilde{g}(\tilde{f}(x)) - x = g(f(x)) - x = 0$
- ✓ Distortion is at most $3D$ → Simple calculation

5 Extension by One Point

Let: $f: A \to \mathbb{R}^m$ be a map (where $A \subset \mathbb{R}^n$) and $u \in \mathbb{R}^n$ be a point

Simplifying assumptions: Assume wlog that

- f is non contracting
- $0 \in A$ and $f(0) = 0$
- $\|u\| = 1$ and 0 is the closest point in A to u

Step 1: find a point $u' \in \mathbb{R}^m$ s.t. → Using Minimax Theorem

- $\|u'\| \le 1$
- Inner products are approximately preserved i.e.,

$$|\langle u', f(v) \rangle - \langle u, v \rangle| \le c\sqrt{\epsilon}(\|v\|^2 + \|u\|^2) \text{ for all } v \in V$$

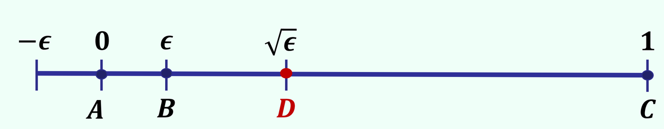
Step 2:

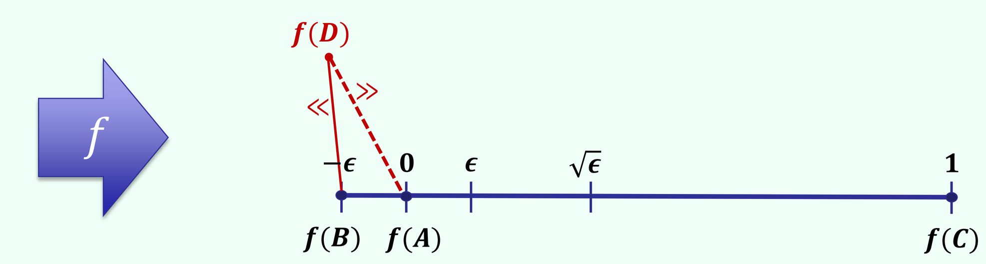
- Let $f(u) = u' \oplus \sqrt{1 - \|u'\|^2}$ which is in $m+1$ dimensions; then

✓ $\|f(u)\|^2 = 1$

✓ $\left| \|f(u) - f(v)\|^2 - \|u - v\|^2 \right| =$
$\left| \|f(u)\|^2 + \|f(v)\|^2 - 2\langle f(u), f(v) \rangle - \|u\|^2 - \|v\|^2 + 2\langle u, v \rangle \right| =$
$O\left(\epsilon(\|u\|^2 + \|v\|^2) + c\sqrt{\epsilon}(\|v\|^2 + \|u\|^2)\right) = O\left(\sqrt{\epsilon}(\|v\|^2 + \|u\|^2)\right) =$
$O(\sqrt{\epsilon}\,\|u - v\|^2)$

Lower Bound: $1 + \Omega(\sqrt{\epsilon})$





6 Extension to the Line

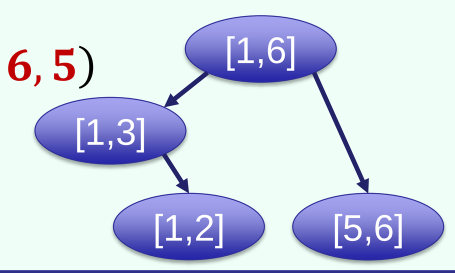
Given: a near isometric map $f: A \to \mathbb{R}$ where $A \subset \mathbb{R}$

- such a map should be very structured.

Permutations: permutation corresponding to the ordering defined by the map

- **Valid:** only if it **excludes (3,1,4,2)** and **(2,4,1,3)** as a "sub-permutation"
- **Flips:** such a permutation can be decomposed into a sequence of **"laminar" flips** (reversing an interval)

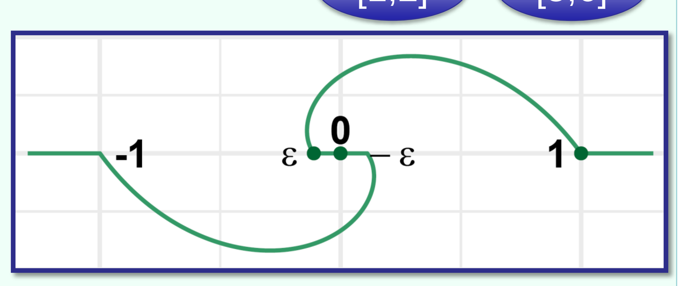
$(1,2,3,4,5,6) \to (3,2,1,4,5,6) \to (3,1,2,4,5,6) \to (3,1,2,4,6,5)$



Spirals:

Basic case: f maps $(0, \epsilon, 1)$ to $(0, -\epsilon, 1)$; extend it to the segment [0,1]

- Map $[0, \epsilon]$ to $[0, -\epsilon]$ linearly
- For $\epsilon \le x \le 1$ map x to $g(x) = (r(x), \phi(x))$ in **polar coordinates**
- $r(x) = x$ and $\phi(x) = \frac{\pi \ln 1/x}{\ln 1/\epsilon}$
- **Distortion** is $1 + O(1/\ln^2(1/\epsilon))$
- This is **tight!**



General case: for each flip, we add a spiral of the "right" scale