Nonlinear Dimension Reduction via Outer Bi-Lipschitz Extensions

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1 Definitions

Lipschitz Extension

- A map \( f : X \to Y \) is \( C \)-Lipschitz if for all \( x, x' \in X \):
  \[ d_Y(f(x), f(x')) \leq C \cdot d_X(x, x') \]

- A Lipschitz extension of a \( C \)-Lipschitz map \( f : A \to Y \), where \( A \subseteq X \), is a \( C' \)-Lipschitz map \( f' : X \to Y \) such that \( f(x) = f'(x) \) for all \( x \in A \).

Kirszbraun theorem: For \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \), there is always an extension with \( C' = C \); every map \( f : A \to \mathbb{R}^m \) can be extended to the whole \( \mathbb{R}^n \) keeping the same Lipschitz constant.

Bi-Lipschitz Extension

- A map \( f : X \to Y \) is \( D \)-bi-Lipschitz or of distortion \( D \) if for some \( \lambda \) and all \( x, x' \in X \):
  \[ \lambda \cdot d_X(x, x') \leq d_Y(f(x), f(x')) \leq D \cdot \lambda \cdot d_X(x, x') \]

Consider Bi-Lipschitz extensions of bi-Lipschitz maps

- Is there an analogue of the Kirszbraun theorem for bi-Lipschitz maps?
  - No direct analogue!
  - Allow \( f' \) to use extra coordinates!

Map \( f' : X \to \mathbb{R}^{m+n} \) is a \( D' \)-outer bi-Lipschitz extension of a \( D \)-bi-Lipschitz map \( f : A \to Y \), where \( A \subseteq X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^m \), if
  - \( f' \) is an outer extension of \( f \): for every \( x \in A \)
    \[ f'(x) = f(x) \oplus (0, \ldots, 0) \]
  - \( f' \) is \( D' \) bi-Lipschitz

2 Results

Consider a \( D \)-bi-Lipschitz map \( f : A \to \mathbb{R}^m \) where \( A \subseteq \mathbb{R}^n \),

Analogue of the Kirszbraun theorem:

- There is an outer bi-Lipschitz extension \( f' : \mathbb{R}^n \to \mathbb{R}^{m+n} \) of \( f \) with distortion 3

Near isometric maps: assume that \( f \) has distortion \( 1 + \epsilon \)

- \( f \) can be extended by one point to \( f' : A \cup \{u\} \to \mathbb{R}^{m+1} \) with distortion \( 1 + O(\sqrt{\epsilon}) \)
  This bound is tight.

- 1-dimensional case: if \( n = m = 1 \), \( f \) can be extended to \( f' : \mathbb{R} \to \mathbb{R}^2 \), with distortion \( 1 + O(1/\log^3 n) \). This bound is tight.

Open Problem: complete the picture for higher dimensions and extensions by more than a single point.

3 Applications to Dimensionality Reduction

Prioritized Johnson-Lindenstrauss

Input:
- a set \( X \) of \( n \) points in \( \mathbb{R}^d \)
- a ranking \( \pi \) on them

Goal: reduce the dimension s.t.
\[ f(x) \in \mathbb{R}^d \cap \mathbb{R}^\log n \]
where \( r \) is the rank of \( x \) and \( g \) is polylogarithmic

Dimensional Reduction

Input: a set \( X \subseteq \mathbb{R}^d \) of \( n \) terminals

Goal: find a map \( f : \mathbb{R}^d \to \mathbb{R}^2 \) s.t. for any \( p \in \mathbb{R}^d \) and any terminal \( x \in X \),
\[ |x - p| \leq |f(x) - f(p)| \leq D \cdot |x - p| \]

4 Analogue of Kirszbraun

Let
- \( f(x) : A \to \mathbb{R}^m \) be our map with distortion \( D \)
- \( g = f' \circ f(A) \to \mathbb{R}^n \) be its inverse
- \( f(x), f(y) \to \mathbb{R}^m \) be the Lipschitz extension of \( f \)
- \( g(x), g(y) \to \mathbb{R}^n \) be the Lipschitz extension of \( g \)

Kirszbraun-like: \( g \) is \( D \)-Lipschitz extension of \( f \)

- Assume \( f \) is non-contracting
- \( 0 \in A \) and \( f(0) = 0 \)
- \( \lambda = 1 \) and \( 0 \) is the closest point in \( A \) to \( u \)

Step 1: find a point \( u^* \) in \( \mathbb{R}^m \) s.t.
Using Minimax Theorem
- \( \lambda u^* \leq 1 \)
- Inner products are approximately preserved i.e., \( \langle u^*, (v) \rangle - \langle u, (v) \rangle \leq c\sqrt{(1 + \|v\|^2)} \) for all \( v \in V \)

Step 2:
- Let \( u = u^* \oplus v \), where \( \|v\| \leq 1 \)
- Use some inner products are preserved i.e., \( \langle u', (v') \rangle - \langle u, (v) \rangle \leq c\sqrt{(1 + \|v\|^2)} \) for all \( v \in V \)

Lower Bound: \( 1 + O(\sqrt{\epsilon}) \)

5 Extension by One Point

Let: \( f : A \to \mathbb{R}^m \) be a map (where \( A \subseteq \mathbb{R}^n \)) and \( u \in \mathbb{R}^n \) be a point

Simplifying assumptions: Assume wlog that
- \( f \) is non-contracting
- \( 0 \in A \) and \( f(0) = 0 \)
- \( \lambda = 1 \) and \( 0 \) is the closest point in \( A \) to \( u \)

Step 1:
- find a point \( u^* \) in \( \mathbb{R}^m \) s.t.
- \( \lambda u^* \leq 1 \)
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Step 2:
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Lower Bound: \( 1 + O(\sqrt{\epsilon}) \)

6 Extension to the Line

Given: a near isometric map \( f : A \to \mathbb{R} \) where \( A \subseteq \mathbb{R} \)
- such a map should be very structured

Permutations: permutation corresponding to the ordering defined by the map
- Valid: only if it excludes \( (3,1,4,2) \) and \( (2,4,1,3) \) as a “sub-permutation”
- Flips: such a permutation can be decomposed into a sequence of “laminar” flips (reversing an interval)
- (1,2,3,4,5,6) \to (3,2,4,1,5,6) \to (3,1,2,4,5,6) \to (3,1,2,4,6,5)

Spirals:

Basic case: \( f \) maps \((0,1)\) to \((0,-\epsilon,1)\); extend it to the segment \([0,1]\)
- Map \([0,\epsilon] \to [0,-\epsilon] \)
- For \( \epsilon \leq s \leq 1 \) map \( x \to g(x) = (r(x), \phi(x)) \) in polar coordinates
  \[ r(x) = x \text{ and } \phi(x) = \frac{\pi}{\ln(1/\epsilon)} \ln(1/\epsilon) \]
  \[ \text{Distortion is } 1 + O(1/\ln^2(1/\epsilon)) \]
  \[ \text{This is tight!} \]

General case: for each flip, we add a spiral of the “right” scale

[Diagram of spirals and dimensionality reduction]