Last time: $\forall n \geq 0$

A $n$-point $(X, d_X)$ is a data structure of size $O(n^{1+\varepsilon})$ s.t. one can estimate $d_X(x_i, x_j)$ up to $O(1/\varepsilon)$ within time $O(1)$.

Today: better approximation, slower query time.

$\textbf{Th}$ $\forall k \geq 1$ can get space $O(kn^{1+\varepsilon/k})$, approximation $2k-1$, query time $O(k)$.

Let's forget about query time and assume that $X$ is induced by an unweighted undirected graph $G = (V, E)$.

**Girth:** $g$, the length of the shortest cycle.

**Cl 1:** Any graph of girth $\geq 2k$ has $O(n^{1+\varepsilon/k})$ edges.

**Cl 2:** There exist graphs of girth $\geq 2k$ with $n^{1+\Omega(1/k)}$ edges. (Known for small $k$).

**PF (Cl 1):** remove all nodes of degree $\leq n^{1/k}$.

The remaining graph if non-empty has girth $\geq 2k$. 
Pf (CL 2)
Probalistic method.
\[ G(n, p) \quad p = n^{\lambda - 1} \quad \lambda < 1. \]
\[ E[\# \text{short cycles}] \leq \sum_{j=3}^{2k} n^{j} p^{j} \leq O(n^{2k\lambda}). \]
if \[ zk \lambda < 1 \Rightarrow \ll \frac{n}{100}. \]
By Markov at most \[ \frac{n}{50} \] cycles whp.
All degrees are \( \Theta(n^{\lambda}) \) whp.
Remove a vertex from each short cycle
\[ O(n^{1+\lambda}) \text{ edges} \quad \lambda < \frac{1}{2k}, \]
\[ O(n^{1+\frac{1}{2k}}) \text{ edges} \]
Greedy spanner:
add edges that don't create cycles of length \[ 2k \]
\[ \leq O(n^{1+1/k}) \text{ edges} \]
\[ (2k-1) - \text{approx.} \]
Any approx. \( \leq 2k-1-\varepsilon \) requires \( \Omega(n^{1+1/k}) \) space
What about fast query time?

\[ k = 2 \quad (3\text{-approx, } n^{3/2}\text{-ish space}) \]

Sample \( O(\sqrt{n}) \) landmarks:

\[
\Pr \left[ x - \text{landmark} \right] = \frac{1}{\sqrt{n}}
\]

For points:

- store distances to all landmarks
- store distances to all points closer than the closest landmark, \( B(v) \).

Query:

if \( u \in B(v) \Rightarrow \text{know the answer } \)

\[
d(u, w) + d(w, v) \leq 3d(u, v).
\]

\[
\leq d(u, v) + d(u, v) + d(v, w) \leq 2d(u, v)
\]

Space: \( n^{3/2} \).
\[ k = 3 \quad \text{5-approx} \quad n^{4/3} \]

\[ n^{1/3} \quad L_2 \]

\[ n^{2/3} \quad L_1 \]

\[ n \quad \text{Query:} \]

if \( u \in B(v) \Rightarrow d(u, v) \)

if \( L_1(v) \in B(u) \Rightarrow d(L_1(v), v) + d(u, L_1(v)) \)

\[ d(u, L_2(u)) + d(L_2(u), v) \]

\[ d(u, L_2(u)) + d(L_2(u), v) \leq \]

\[ d(u, L_1(v)) \leq d(u, L_2(u)) + d(u, v) \leq 3d(u, v) \leq 2d(u, v) \]