Next several lectures: graph partitioning.

Today and next time:
"as many edges between parts as possible."

After that:
"as few"

Common theme:
1) Embed a graph into $\mathbb{R}^d$ or other geometric space
2) Use geometric partitioning.

Tools: SDPs, spectral graph theory, metric embeddings, LSH-like partitions.

**MAX-CUT**

$G = (V, E)$, $|V| = n$, $|E| = m$

Find a partition $V = V_1 \cup V_2$ s.t. $E(V_1, V_2) \rightarrow \max.$ (denote $\text{OPT}$)

\[
\text{OPT} \geq \frac{m}{2}.
\]

Find a poly-time $\frac{1}{2}$-approx. algorithm
There exists a poly-time \( 0.87856... \)-approximation algorithm for MAX-CUT.

1) \( \text{OPT} = \max_{x: V \to \{-1, 1\}} \sum_{(u, v) \in E} \frac{1 - x_u x_v}{2} \)

\( \mathcal{NP} \)-hard.

2) Idea: instead of searching for \( x: V \to \{-1, 1\} \), search for \( x: V \to S^{n-1} \).

\( \text{OPT}' = \max_{x: V \to S^{n-1}} \sum_{(u, v) \in E} \frac{1 - \langle x_u, x_v \rangle}{2} \)

\( \mathcal{CL} \) \( \text{OPT} \leq \text{OPT}' \)

\( \mathcal{CL} \) \( \text{OPT}' \) and the respective embedding \( x: V \to S^{n-1} \) can be computed in polynomial time.

Semidefinite programming (SDP).

Look at the Gram matrix \( M_{uv} = \langle x_u, x_v \rangle \).
$M \succ 0$

$(\forall y \ y^T M y \succ 0) \mathrm{ M=Symmm}$

$\forall M \succ 0 \ \exists x_u : V \rightarrow \mathbb{R}^n \ s.t.
M_{uv} = \langle x_u , x_v \rangle$

Cholesky decomposition $M = A A^t$. PSD symmetric matrices form a convex cone $L$. One can efficiently optimize over it! In theory: Ellipsoid method.

Overall program for $OPT'$

$$OPT' = \max \sum \frac{1 - M_{uv}}{Med : (u,v) \in V \quad 2 \quad V^v M_{uv} = 1}$$

Cholesky decomposition gives an embedding $x : V \rightarrow S^n$

Can't lower the dimension in general

So, we have an embedding. What should one do next?
Answer: use random partitions.

Let's find a cut using a random hyperplane:

Intuition: if \( \langle x_u, x_v \rangle \) contributes a lot, then often at a different sides.

\[
E[\text{found cut}] = \sum_{(u,v) \in E} \Pr[(u,v) \text{ is cut}] = \\
\geq \frac{1}{\pi} \sum_{(u,v) \in E} \frac{\angle (x_u, x_v)}{2} \\
\geq \frac{1}{\pi} \cdot \text{???.} \cdot \sum_{(u,v) \in E} \frac{1 - \langle x_u, x_v \rangle}{2}
\]

\[
C = \inf_{x_u, x_v \in S^{d-1}} \frac{2 \angle (x_u, x_v)}{1 - \langle x_u, x_v \rangle} = \\
= \inf_{0 < \gamma < \pi} \frac{2 \gamma}{1 - \cos \gamma}
\]

\[
E[\text{found cut}] \geq \frac{2}{\pi} \cdot \inf_{0 < \gamma < \pi} \frac{\gamma}{1 - \cos \gamma} \cdot \text{OPT', ???.}
\]
\[ \frac{2}{\pi} \inf_{0 < y < \pi} \frac{y}{1 - \cos y} \]

\[ y = \text{small} \Rightarrow +\infty \]
\[ y = \frac{\pi}{2} \Rightarrow 1 \]
\[ y = 2.33412... \Rightarrow 0.87856... \]
\[ y = \pi \Rightarrow 1 \]

Can we do better than 0.87856...?

The assumption "Unique Games Conjecture" (which does not contain any magic number) it is NP-hard to approximate MAX-CUT better than within a factor:
\[ \frac{2}{\pi} \inf_{0 < y < \pi} \frac{y}{1 - \cos y} \]

0.87856... is a property of nature not human brain.

Coloring
A graph \( G = (V, E) \) is \( k \)-colorable if
\[ \exists f : V \to [k] \text{ s.t. } \forall \text{ edge } (u,v) \in E \]
\[ f(u) \neq f(v). \]
Given a 3-colorable graph

Two parametrizations:
- Color into 3 colors to have as few as possible monochromatic edges (homework).
- Color fully correctly into as few colors as possible (now).

Claim: Can always get \( n \) colors.

\[ \Delta \text{- maximum degree.} \]

Claim: Can get \( \Delta + 1 \) colors.

Never used 3-colorability.

Theorem: Can get \( O(\sqrt{n}) \) colors.

Proof: Case 1: \( \exists v \text{ } \deg v > \sqrt{n} \)
- Color \( v \) and its neighbors into 3 fresh colors, recurse on the remainder.

Case 2: \( \Delta \leq \sqrt{n} \)
- Color into \( \Delta + 1 \) colors.

Will use SDPs to achieve \( \tilde{O}(n^{1/4}) \) colors.
Plan: 
1) \( \tilde{O}(\Delta^{1/3}) \) colors
   
2) Use a trick similar to \( O(\sqrt{n}) \) to get \( \tilde{O}(n^{1/4}) \) colors.

SDP relaxation
\[ x: V \rightarrow S^{d-1} \]
\[ \forall e = (u, v) \in E \quad \langle x_u, x_v \rangle \leq -\frac{1}{2} \quad (\geq \frac{2\pi}{3}) \]

CL: Such embedding exists \( \forall 3 \)-colorable \( G \) and can be found in poly-time.

How to extract a good coloring?
Need to find large independent set.

Idea: pick a random set \( \tilde{V} \subset V \)
Contains an independent set of size \( \tilde{\Omega}(\frac{n}{\Delta^{1/3}}) \).

Construction: \( \eta > 0 \quad g \sim N(0, 1)^d \)
\[ \tilde{V} = \{ v \in V \mid \langle x_v, g \rangle \geq \eta \} \]
Choice of \( \eta \) and analysis \( \Rightarrow \) next time.