COMS W6998-5: Algorithms through Geometric Lens (Fall'17) Oct 25, 2017

Lecture 15 – Linear Algebraic Notion of Graph Connectivity

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1 Introduction

From last lecture:

Remark 1.

$$\forall x \neq 0, |R(x)| \leq 1 \Rightarrow 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$$

Remark 2.

$$R(x) = 1 \iff \forall (i,j) \in E, \alpha = \pm 1, \frac{v_i}{\sqrt{d_i}} = \alpha \frac{v_j}{\sqrt{d_j}}$$

Lemma 3. $\lambda_2 < 1 \iff$ *G is connected*

2 Connectivity in a graph

If G has c components then $\lambda_1 = \lambda_2 = \ldots = \lambda_c = 1$

Proof. (By contradiction) Assume $\lambda_2 = 1$, then

$$\exists v_2 = \arg\max_{v' \perp v} R(v_2')$$

$R(v_2') = 1$. From last lecture, $v_1 = (\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$ is the eigenvector corresponding to λ_1. By remark $\Rightarrow \forall (i,j) \in E, \frac{v_i'}{\sqrt{d_i}} = \frac{v_j'}{\sqrt{d_j}}$
If G is connected, then

$$\exists \beta, \frac{v_i'}{\sqrt{d_i}} = \beta \quad \forall i \in [n]$$

$$\Rightarrow v' = (\beta\sqrt{d_1}, \beta\sqrt{d_2}, \ldots, \beta\sqrt{d_n})$$

$$0 = v' \cdot v = \sum_i \beta d_i \Rightarrow \beta = 0 \Rightarrow \textit{contradiction}$$

□

We want to be able to say more about the connectivity of the graph based on the value of λ_2, i.e. how sparse or dense it is.

3 Linear algebraic notion of connectivity

Laplacian of a graph G

$$\hat{L} = I - \hat{A} \quad \text{(normalized)}$$

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \text{ are eigenvalues of } \hat{L}$$

Fact 4. $\mu_i = 1 - \lambda_i$ *with eigenvectors being exactly the same*

This immediately implies that

$$\mu_1 = 0$$

$$\mu_2 = 1 - \lambda_2 \quad (\leq 2)$$

μ_2 is linear algebraic notion of connectivity

$$\hat{L} = I - \hat{A} = D^{-\frac{1}{2}} D D^{-\frac{1}{2}} - D^{-\frac{1}{2}} A D^{-\frac{1}{2}} = D^{-\frac{1}{2}} \underbrace{(D - A)}_{\text{Laplacian of } G} D^{-\frac{1}{2}}$$

$$L = \begin{bmatrix} d1 & & & \\ & d2 & & \\ & l_{ij} & \ddots & \\ & & & dn \end{bmatrix}, \text{ where } l_{ij} = \begin{cases} -1, & (ij) \in E \\ 0, & oth \end{cases} \tag{1}$$

Property 5. *fix any* $x \in \mathbb{R}^n$,

$$x^T L x = \sum_{(ij) \in E,\ i<j} (x_i - x_j)^2$$

Proof.

$$L = \sum_{(ij) \in E,\ i<j} L_{ij} \text{ , where } L_{ij} \text{ is Laplacian of graph with precisely one edge } (ij)$$

$$L_{ij} = \begin{bmatrix} +1 - 1 \\ -1 + 1 \end{bmatrix}$$

$$x^T L x = x^T \left(\sum_{(ij) \in E,\ i<j} L_{ij} \right) x = \sum_{(ij) \in E,\ i<j} x^T L_{ij} x = \sum_{(ij) \in E,\ i<j} x_i^2 + x_j^2 - 2x_i x_j =$$

$$= \sum_{(ij) \in E,\ i<j} (x_i - x_j)^2$$

□

Now,
$S \subset [n] \to$ cut in G
$\bar{S} = [n] \backslash S$
$\text{cut}(S) = \#$ edges crossing S and $\bar{S}$

Fact 6. *let* $x = 1_S$*, where* $\left((1_S)_i = \begin{cases} 1, \text{ if } i \in S \\ 0, oth \end{cases}\right)$*, then* $x^T Lx = cut(S)$

Proof. $x^T Lx = \sum_{(ij)\in E,\ i<j} (x_i - x_j)^2 = \text{cut}(S)$ □

G is connected if there is no particularly small cut on G.

Definition 7. $\delta(S) = cut(S)$

Definition 8. $Vol(S) = \sum_{i\in S} d_i$

Definition 9 (conductance)**.** $\phi(S) = \frac{\delta(S)}{Vol(S)} \quad (\leq 1)$

Conductance of G,

$$\phi(G) = \min_{S\subset[n]} \phi(S) = \min_{S\subset[n]} \frac{\delta(S)}{\min\{\text{Vol}(S), \text{Vol}(\bar{S})\}}$$

$$\text{Vol}(S) \leq \text{Vol}(G) \cdot \frac{1}{2}$$

Theorem 10 (Cheeger inequality)**.**

$$\frac{\mu_2}{2} \leq \phi(G) \leq \sqrt{(}2\mu_2)$$

Fact 11. *computing* $\phi(G)$ *is NP-hard*

We can approximate by calculating μ_2. Cheeger inequality gives approximation algorithm for $\phi(G)$ and approximation ratio is:

$$\frac{\phi(G)^2}{2} \leq \mu_2 \leq 2\phi(G)$$

$$\frac{4}{\phi(G)}$$

$\phi(G)$ is large, then a good approximation; $\phi(G)$ is small, then a very bad approximation.
If G is poorly connected, then $\mu_2 \to 0 \quad (\mu_2 = 1 - \lambda_2)$
To prove that

$$\frac{\mu_2}{2} \leq \phi(G) = \min_S \phi(S),$$

it suffices to prove

$$\forall S \neq 0, \text{Vol}(S) \leq \frac{1}{2}\text{Vol}(G), \frac{\mu_2}{2} \leq \phi(S)$$

Proof.

$$\mu_2 = \min_{v_2 \perp v_1} \frac{R(v_2)}{\|v_2\|_2^2} = \min \frac{v_2^T \hat{L} v_2}{\|v_2\|_2^2}$$

Fix S, enough to construct $x \in \mathbb{R}^n$, $x \perp v_1$, and

$$\frac{x^T \hat{L} x}{\|x\|_2^2} \leq 2\phi(S)$$

$$\frac{x^T \hat{L} x}{\|x\|_2^2} = \frac{x^T D^{-\frac{1}{2}} L D^{-\frac{1}{2}} x}{x^T x} = \frac{y^T L y}{y^T D^{\frac{1}{2}} D^{\frac{1}{2}} y} = \frac{y^T L y}{y^T D y}$$

$$y = 1_S \Rightarrow \frac{y^T L y}{y^T D y} = \frac{\delta(S)}{\text{Vol}(S)} = \phi(S)$$

Issue: $x \not\perp v_1 = (\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$, where $x = D^{\frac{1}{2}} y$.
We can orthonormalize x w.r.t. v_1:

$$x = D^{\frac{1}{2}} y - \sigma v_1 \Rightarrow y = D^{-\frac{1}{2}} x = 1_S - \sigma D^{-\frac{1}{2}} v_1$$

$$\langle D^{\frac{1}{2}} y, v_1 \rangle = y^T D^{\frac{1}{2}} v_1 = (1_S - \sigma D^{-\frac{1}{2}} v_1)^T D^{\frac{1}{2}} v_1 = 1_S D^{\frac{1}{2}} v_1 - \sigma \|D^{-\frac{1}{2}} v_1\|_2^2 = 0$$

$$\sigma = \frac{1}{\|D^{-\frac{1}{2}} v_1\|_2^2} \sum_{i \in S} \sqrt{d_i} \sqrt{d_i} = \frac{vol(S)}{vol(G)}$$

To be continued... $\square$