COMS 6998-5: Algorithms Through Geometric Lens — Oct 23, 2017

Lecture 14 – Spectral Graph Theory

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1 Introduction

In the last lecture, we introduced Spectral Graph Theory and the idea to examine the eigenvalues and their corresponding eigenvectors to deduce combinatorial properties of a graph. We looked at the diffusion operation and we defined the Rayleigh Quotient. In this lecture, we show that properties of the first and second eigenvalues of an adjacency matrix tell us about the connectivity properties of the corresponding undirected graph.

2 Symmetric Matrix Transformation

Let X_0 be the initial distribution of weights in the graph. We saw last lecture that

$$X_1 = AD^{-1}X_0$$

and

$$X_t = (AD^{-1})^t X_0$$

Note that (AD^{-1}) here is no longer a symmetric matrix. We would like to make this expression cleaner by writing it as a power of a symmetric matrix.

Definition 1. $\hat{A} := D^{-1/2}AD^{1/2}$
We have that $X_{t+1} = AD^{-1}X_t$, *so* $D^{-1/2}X_{t+1} = D^{-1/2}AD^{-1}X_t = D^{-1/2}X_{t+1} = (D^{-1/2}AD^{-1/2})(D^{-1/2}X_t)$.

Definition 2. $Y_t = D^{-1/2}X_t$

Definition 3. $\hat{A}_{i,j} = \frac{A_{i,j}}{\sqrt{d_i}\sqrt{d_j}}$

From the definitions above, we have:

$$Y_{t+1} = \hat{A}Y_t = \hat{A}^t Y_0$$

At the stationary point X_t, we have:

$$X_t = X_{t+1} = (AD^{-1})X_t$$

or

$$\hat{A}Y_t = Y_t$$

which is the eigenvector corresponding to the eigenvalue $\lambda = 1$.

3 Properties of the graph based on Eigenvalues

Suppose we fix $Y_0 = \Sigma \alpha_i v_i$, where v_i is the i'th orthonormal eigenvector of $\hat{A}$. Note that we can do this by the Spectral Decomposition Theorem.

$$Y_t = \hat{A}^t Y_0 = (\Sigma \lambda_i^t v_i v_i^T)(\Sigma \alpha_i v_i) = \Sigma_{i=1}^n \lambda_i^t \alpha_i v_i$$

As $t \to \infty$, if $\lambda_i < 0, \lambda_i^t \to 0$ and if $\lambda_i > 0, \lambda_i^t \to \infty$.
$|Y_t|^2 = \Sigma \alpha_i^2 \lambda_i^{2t}$, so we intuitively expect that $|\lambda_i| \leq 1$.

Lemma 4. $\lambda_1 = 1$

Proof. First let's prove that $\lambda_1 \geq 1$. We assume the ordering $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We know that

$$\lambda_1 = max_{x \neq 0} R(x)$$

from the previous lecture. If we can show that there exists an x such that $R(x) = 1$, then we know that λ_1 has to be at least 1. Indeed, this is true for $x = v = (\sqrt{d_1}, \sqrt{d_2}, \ldots \sqrt{d_n})$ where d_i is the degree of the ith node.

$$R(v) = \frac{v^T \hat{A} v}{v^T v} = \frac{v^T D^{-1/2} A D^{-1/2} v}{d_1 + d_2 + \ldots + d_n} = \frac{\mathbb{1}_n^T A \mathbb{1}_n}{d_1 + \ldots + d_n} = \frac{\mathbb{1}_n^T [d_1, \ldots, d_n]^T}{d_1 + d_2 + \ldots + d_n} = 1$$

And so $\lambda_1 = max_{v \neq 0} R(v) \geq 1$.
Now we show $\lambda_1 \leq 1$.

$$R(v) = \frac{v^T \hat{A} v}{v^T v} = \frac{\sum_{(i,j) \in E} v_i \hat{A}_{ij} v_j}{||v||_2^2} = \frac{\sum_{(i,j) \in E} \frac{1}{\sqrt{d_i d_j}} v_i v_j}{||v||_2^2}$$

Using Cauchy-Schwartz, we get:

$$\frac{\sum_{(i,j) \in E} \frac{1}{\sqrt{d_i d_j}} v_i v_j}{||v||_2^2} \leq \frac{(\sum_{(i,j) \in E} (\frac{v_i}{\sqrt{d_i}})^2)^{1/2} (\sum_{(i,j) \in E} (\frac{v_j}{\sqrt{d_j}})^2)^{1/2}}{||v||_2^2} = \frac{(\sum_i \frac{v_i^2 d_i}{d_i})^{1/2} (\sum_j \frac{v_j^2 d_j}{d_j})^{1/2}}{||v||_2^2} = \frac{(\sum_i v_i^2)^{1/2} (\sum_j v_j^2)^{1/2}}{||v||_2^2} =$$

as required. Hence, $R(v) \leq 1 \forall v$. So, $\lambda_1 = 1$ since we have shown $\lambda_1 \geq 1$ before.
Note that if $R(v) = 1$, then by the condition for equality in the inequality, we must have $\frac{v_i}{\sqrt{d_i}} = \alpha \frac{v_j}{\sqrt{d_j}}$ for some constant α.
Note that since both summations are identical, we have a j term on the left and an i term on the right so we must have: $\frac{v_j}{\sqrt{d_j}} = \alpha \frac{v_i}{\sqrt{d_i}}$. This forces $\alpha^2 = 1$ or $\alpha = \pm 1$. This gives us intuition to why we picked $(\sqrt{d_1}, \ldots \sqrt{d_n})$ in the first part of the proof. Also note that this must be the eigenvector corresponding to $\lambda = 1$. So the stationary point is:

$$X^* = D^{1/2} Y^* = (d_1, \ldots, d_n)$$

We can multiply eigenvectors by any constant, so to get a stationary distribution, we use:

$$X^*_{dist} = (\frac{d_1}{\sum d_i}, \ldots, \frac{d_n}{\sum d_i})$$

□

Claim 5. *If $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$ are the eigenvalues of $\hat{A}$ and $v_1 \ldots v_n$ are the eigenvectors, then $\lambda_1^t, \lambda_2^t, \ldots, \lambda_n^t$ are the eigenvalues of $\hat{A}^t$*

Proof. Suppose the claim is true for $t = 1, ..., k-1$ (inductive hypothesis). The base cases are easy to show. Then,

$$\hat{A}^k v_i = \hat{A}^{k-1}\hat{A}v_i = \hat{A}^{k-1}\lambda_i v_i = \lambda_i^k v_i$$

Hence, our induction is complete and the claim holds $\forall t \in \mathbb{N}$. $\square$

Lemma 6. $\lambda_2 < 1 \iff G$ *is connected.*

Proof. Suppose G is disconnected and has two components. Then, its vertex set can be separated into parts $\{1, 2, ..., k\}$ and $\{k+1, ..., n\}$ such that vertices with the respective indices in these two sets are disjoint and not connected by an edge. Now, consider the vectors:

$$v_1 = (\sqrt{d_1}, ..., \sqrt{d_k}, 0, ..., 0), v_2 = (0, ..., 0, \sqrt{d_{k+1}}, ..., \sqrt{d_n})$$

The corresponding matrix $\hat{A}$ for the disconnected graph looks like:

$$\left[\begin{array}{c|c} \hat{A}_{1..k} & 0 \\ \hline 0 & \hat{A}_{k+1...n} \end{array}\right]$$

Note that:

$$R(v_1) = R(v_2) = 1 \implies \lambda_1 = \lambda_2 = 1$$

Note that v_1 and v_2 are orthogonal to each other, which implies that $\lambda_2 = 1$, by the proof of Lemma 1.
Next time: To show that G is connected $\implies \lambda_2 < 1$. $\square$